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On the cohomology of Frobenius algebras II*
Crossed products

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Introduction

This paper is a continuation of the previous paper [7]. Let R be a commutative ring. Let Γ be a Frobenius algebra over R and Λ be a Frobenius R-subalgebra of Γ such that the extension of rings Γ/Λ is a Frobenius extension. The purpose of this paper is to give certain relations between the complete cohomology group in the sense of Nakayama [6] of Γ and that of Λ, along the cohomology theory of finite groups as in [2, 3, 5, 8]. More precisely, to reduce the calculation of $H^r(\Gamma, -)$ to that of $H^r(\Lambda, -)$ by defining a *restriction map*, a *corestriction map*, and in particular a *conjugation map* when Γ is a crossed product over Λ.

In Section 1, we will show some general facts for a Frobenius extension Γ of a Frobenius R-algebra Λ, and we will define the restriction map $\mathrm{Res}: H^r(\Gamma, M) \to H^r(\Lambda, M)$ and the corestriction map $\mathrm{Cor}: H^r(\Lambda, M) \to H^r(\Gamma, M)$ for any two-sided Γ-module M. We will prove some fundamental properties of them (Lemma 1.2, Lemma 1.3). In Section 2, let Λ be a commutative Frobenius R-algebra with a finite group G of automorphisms of Λ over R. Suppose that Γ is a crossed product over Λ with any normalized factor set; $\Gamma = \sum_{\sigma \in G} \oplus \Lambda w_\sigma$. Then we define the conjugation map $\gamma_\tau : H^r(\Lambda, M) \to H^r(\Lambda, M)$ for $\tau \in G$ and for any two-sided Γ-module M, and prove some fundamental properties (Lemma 2.1, Lemma 2.2). In Section 2.2, we describe the main results using the above materials under certain conditions with $R = \mathbb{Z}$, that

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is, for a prime integer l such that $l \mid N_\Gamma(\Gamma)$ and $l \nmid |G|$, we have $\mathrm{Res}(H^r(\Gamma, M)_{(l)}) = [H^r(\Lambda, M)_{(l)}]^G$ (Proposition 2.3), and then $H^*(\Lambda, \Gamma)_{(l)} = \sum_{r\in\mathbb{Z}} \bigoplus H^r(\Lambda, \Gamma)_{(l)}$ has an invertible element of nonzero degree if and only if so does $H^*(\Gamma, \Gamma)_{(l)} = \sum_{r\in\mathbb{Z}} \bigoplus H^r(\Gamma, \Gamma)_{(l)}$ (Theorem 2.4). In Section 3, as applications of Theorem 2.4 and [7, Proposition 3.8], we will investigate the periodicity of the cohomology groups of crossed products which appear as $\mathbb{Z}$-orders in a quaternion $\mathbb{Q}$-algebra (Proposition 3.1, Proposition 3.2).

We will maintain the notation and terminology of [7].

1. Restriction and corestriction on the complete cohomology

1.1. Frobenius extensions of Frobenius algebras

Let Γ/Λ be a Frobenius extension of rings of free rank m (see [4]); Γ has a right Λ-basis (a_i) and a left Λ-basis (b_i):

$$\Gamma = a_1\Lambda \oplus \cdots \oplus a_m\Lambda = \Lambda b_1 \oplus \cdots \oplus \Lambda b_m ,$$

which satisfy

$$xa_i = \sum_{j=1}^{m} a_j\beta_{ji}(x) , \qquad b_jx = \sum_{i=1}^{m} \beta_{ji}(x)b_i$$

for every $x \in \Gamma$ and some $\beta_{ji}(x) \in \Lambda$. In such a case, we call a pair $(a_i), (b_i)$ $(1 \le i \le m)$ the *dual bases* of Γ/Λ. Then we have a left Γ-, right Λ-module isomorphism:

$$\phi_{\Gamma/\Lambda} : {}_\Gamma\Gamma_\Lambda \xrightarrow{\sim} \mathrm{Hom}_{\Lambda,-}(\Gamma_\Gamma, \Lambda_\Lambda)$$

by setting $\phi_{\Gamma/\Lambda}(a_i)(b_j) = \delta_{ij}$, and also we have a left Λ-, right Γ-module isomorphism:

$$\phi'_{\Gamma/\Lambda} : {}_\Lambda\Gamma_\Gamma \xrightarrow{\sim} \mathrm{Hom}_{-,\Lambda}({}_\Gamma\Gamma, {}_\Lambda\Lambda)$$

by setting $\phi'_{\Gamma/\Lambda}(b_j)(a_i) = \delta_{ij}$.

Put $\mu_{\Gamma/\Lambda} = \phi_{\Gamma/\Lambda}(1)$ and $N_{\Gamma/\Lambda}(x) = \sum_{i=1}^{m} a_ixb_i$ for $x \in \Gamma$. Then it is easy to see that $\mu_{\Gamma/\Lambda}$ is a two-sided Λ-module homomorphism and

$$x = \sum_{i=1}^{m} \mu_{\Gamma/\Lambda}(xa_i)b_i = \sum_{j=1}^{m} a_j\mu_{\Gamma/\Lambda}(b_jx)$$

for $x \in \Gamma$. In addition, we assume that Λ is a Frobenius algebra over a commutative ring R of free rank n (i.e. the extension of rings Λ/R is a Frobenius extension in the above sense); that is, Λ has the R-dual bases $(u_i), (v_i)$ $(1 \leq i \leq n)$ which satisfies

$$yu_i = \sum_{j=1}^{n} u_j \alpha_{ji}(y), \qquad v_j y = \sum_{i=1}^{n} \alpha_{ji}(y) v_i$$

for every $y \in \Lambda$ and some $\alpha_{ji}(y) \in R$. Then, similarly as in the above, we have a left Λ-isomorphism

$$\phi_\Lambda : {}_\Lambda\Lambda \xrightarrow{\sim} \Lambda^* = \mathrm{Hom}_R(\Lambda_\Lambda, R)$$

by setting $\phi_\Lambda(u_i)(v_j) = \delta_{ij}$. If we put $\mu_\Lambda = \phi_\Lambda(1)$, then

$$y = \sum_{i=1}^{n} \mu_\Lambda(yu_i)v_i = \sum_{j=1}^{n} u_j \mu_\Lambda(v_j y)$$

holds for $y \in \Lambda$. And we put $N_\Lambda(y) = \sum_{i=1}^{n} u_i y v_i$ and $y^{\triangle'} = \sum_{i=1}^{n} \mu_\Lambda(u_i y) v_i$. We call N_Λ and ${}^{\triangle'}$ the *norm* and the *Nakayama automorphism* of Λ/R, respectively. Then Γ has the R-dual bases $(a_i u_j), (v_j b_i)$ $(1 \leq i \leq m, 1 \leq j \leq n)$, which satisfies

$$xa_i u_j = \sum_{k=1}^{m} \sum_{l=1}^{n} a_k u_l \beta_{lj}(\alpha_{ki}(x)), \qquad v_l b_k x = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta_{lj}(\alpha_{ki}(x)) v_j b_i$$

for $x \in \Gamma$. This says that Γ is a Frobenius R-algebra of free rank mn. And then we have a left Γ-isomorphism

$$\phi_\Gamma : {}_\Gamma\Gamma \xrightarrow{\sim} \mathrm{Hom}_R(\Gamma_\Gamma, R)$$

by setting $\phi_\Gamma(a_i u_j)(v_l b_k) = \delta_{(i,j),(k,l)}$. Similarly as in the above, if we put $\mu_\Gamma = \phi_\Gamma(1)$, then

$$x = \sum_{i=1}^{m} \sum_{j=1}^{n} \mu_\Gamma(xa_i u_j) v_j b_i = \sum_{k=1}^{m} \sum_{l=1}^{n} a_k u_l \mu_\Gamma(v_l b_k x)$$

holds for $x \in \Gamma$. In such a case, the norm and the Nakayama automorphism of Γ/R are given by

$$N_\Gamma(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i u_j x v_j b_i, \qquad x^{\triangle} = \sum_{i=1}^{m} \sum_{j=1}^{n} \mu_\Gamma(a_i u_j x) v_j b_i$$

for $x \in \Gamma$, respectively. ${}^{\triangledown}$ denotes the inverse automorphism of ${}^{\triangle}$. Furthermore, it is easily verified that

$$N_{\Gamma/\Lambda} \cdot N_\Lambda = N_\Gamma, \quad \mu_\Lambda \cdot \mu_{\Gamma/\Lambda} = \mu_\Gamma, \quad {}^{\triangle}|_\Lambda = {}^{\triangle'}.$$

1.2. Restriction and corestriction

Notation and assumption being as in Section 1.1, for simplicity, we put

$$\Sigma = \Gamma \otimes_R \Gamma^{\circ}, \quad \Omega = \Lambda \otimes_R \Lambda^{\circ}, \quad \Phi = \Lambda \otimes_R \Gamma^{\circ},$$

where Γ° and Λ° denote the opposite rings of Γ and Λ, respectively. And putting

$$(X_\Gamma)_p = \Gamma \otimes_R \cdots \otimes_R \Gamma \quad ((p+2) \text{ copies of } \Gamma),$$
$$(X_\Lambda)_p = \Lambda \otimes_R \cdots \otimes_R \Lambda \quad ((p+2) \text{ copies of } \Lambda)$$

for $p \geq 0$, (X_Γ, d) and (X_Λ, d) denote the canonical Σ-free resolution of Γ and the canonical Ω-free resolution of Λ as in [7, Section 1.1], respectively. For simplicity, an element $x_0 \otimes x_1 \otimes \cdots \otimes x_p \otimes x_{p+1} \in (X_\Gamma)_p$ is denoted by $x_0[x_1, \ldots, x_p]x_{p+1}$, and so on. Let M be a left Σ-module. Note that M is also regarded as a left Ω- and a left Φ-module. Then we have the following chain complexes which give the cohomology group $H^p(\Gamma, M)$ (resp. $H^p(\Lambda, M)$) for $p \geq 0$ and the modified homology group associated to the Nakayama automorphism $H_q^{\triangle}(\Gamma, M)$ (resp. $H_q^{\triangle'}(\Lambda, M)$) for $q \geq 0$ of Γ (resp. Λ):

$$\cdots \longleftarrow \operatorname{Hom}_\Sigma((X_\Gamma)_1, M) \xleftarrow{d_1^\#} \operatorname{Hom}_\Sigma((X_\Gamma)_0, M) = M$$
$$\xleftarrow[N_\Gamma]{} M = (X_\Gamma)_0^{\triangle} \otimes_\Sigma M \xleftarrow{d_1 \otimes \iota} (X_\Gamma)_1^{\triangle} \otimes_\Sigma M \longleftarrow \cdots,$$
$$\cdots \longleftarrow \operatorname{Hom}_\Omega((X_\Lambda)_1, M) \xleftarrow{d_1^\#} \operatorname{Hom}_\Omega((X_\Lambda)_0, M) = M$$
$$\xleftarrow[N_\Lambda]{} M = (X_\Lambda)_0^{\triangle'} \otimes_\Omega M \xleftarrow{d_1 \otimes \iota} (X_\Lambda)_1^{\triangle'} \otimes_\Omega M \longleftarrow \cdots.$$

Remark 1. By means of the complete cohomology and the complete modified homology groups, we may have the identifications: $H^r(\Gamma, M) \simeq H_{-r-1}^{\triangle}(\Gamma, M)$, $H^r(\Lambda, M) \simeq H_{-r-1}^{\triangle'}(\Lambda, M)$ for every $r \in \mathbb{Z}$ (see [6, Section 3] and [7, Lemma 1.1]).

Now, as a left Φ-free resolution of Γ, we can use (X_Γ, d) and

$$\cdots \to (X_\Lambda)_{p-1} \otimes \Gamma \xrightarrow{d_p} (X_\Lambda)_{p-2} \otimes \Gamma \to \cdots \to \Lambda \otimes \Gamma \xrightarrow{\varepsilon} \Gamma \to 0,$$

and, as a right Φ-free resolution of Γ, we can use (X_Γ, d) and

$$\cdots \to \Gamma \otimes (X_\Lambda)_{p-1} \xrightarrow{d_p} \Gamma \otimes (X_\Lambda)_{p-2} \to \cdots \to \Gamma \otimes \Lambda \xrightarrow{\varepsilon} \Gamma \to 0.$$

We define the left Φ-module homomorphism:

$$\tau_p : (X_\Gamma)_p \to (X_\Lambda)_{p-1} \otimes \Gamma ,$$

$$y_0[y_1, \ldots, y_p]y_{p+1} \mapsto \sum_{i_1,\ldots,i_{p+1}=1}^{m} \mu_{\Gamma/\Lambda}(y_0 a_{i_1})$$
$$\times [\mu_{\Gamma/\Lambda}(b_{i_1}y_1a_{i_2}), \ldots, \mu_{\Gamma/\Lambda}(b_{i_p}y_pa_{i_{p+1}})]b_{i_{p+1}}y_{p+1} ,$$

and we also define the right Φ-module homomorphism:

$$\tau'_q : (X_\Gamma)_q^\triangle \to (\Gamma \otimes (X_\Lambda)_{q-1})^\triangle ,$$

$$y_0[y_1, \ldots, y_q]y_{q+1} \mapsto \sum_{i_1,\ldots,i_{q+1}=1}^{m} y_0 a_{i_1}$$
$$\times [\mu_{\Gamma/\Lambda}(b_{i_1}y_1a_{i_2}), \ldots, \mu_{\Gamma/\Lambda}(b_{i_q}y_qa_{i_{q+1}})]\mu_{\Gamma/\Lambda}(b_{i_{q+1}}y_{q+1}) ,$$

where both sides of the above sequence are regarded as right Φ-modules by

$$w \cdot (x \otimes y^{\mathrm{o}}) = y^\nabla wx \tag{1.1}$$

for $x \otimes y^{\mathrm{o}} \in \Phi$, $w \in (X_\Gamma)_q$ or $w \in \Gamma \otimes (X_\Lambda)_{q-1}$. Then it is easy to see that τ and τ' are chain transformations lifting the identity maps $\Gamma \xrightarrow{=} \Gamma$ and $\Gamma^\triangle \xrightarrow{=} \Gamma^\triangle$, respectively. Thus we have the following lemma:

Lemma 1.1. *The isomorphism* $\mathrm{Ext}^p_\Phi(\Gamma, M) \xrightarrow{\sim} H^p(\Lambda, M)$ *for* $p \geq 1$ *is induced from the homomorphisms on the cochain level*:

$$\mathrm{Hom}_\Phi((X_\Gamma)_p, M) \underset{\tau_p^\#}{\rightleftarrows} \mathrm{Hom}_\Phi((X_\Lambda)_{p-1} \otimes \Gamma, M) \simeq \mathrm{Hom}_\Omega((X_\Lambda)_p, M) , \tag{1.2}$$

where the above right-arrow denotes a natural homomorphism and $\tau_p^\#$ *denotes the map induced from* τ_p. *The isomorphism* $\mathrm{Tor}_q^{\triangle\Phi}(\Gamma, M) \xrightarrow{\sim} H_q^{\triangle'}(\Lambda, M)$ *for* $q \geq 1$ *is induced from the homomorphisms on the chain level*:

$$(X_\Gamma)_q^\triangle \otimes_\Phi M \overset{\tau_q'^\#}{\rightleftarrows} (\Gamma \otimes (X_\Lambda)_{q-1})^\triangle \otimes_\Phi M \simeq (X_\Lambda)_q^{\triangle'} \otimes_\Omega M , \tag{1.3}$$

where the above left-arrow denotes a natural homomorphism and $\tau_q'^\#$ *denotes the map induced from* τ'_q, *where* $\mathrm{Tor}_q^{\triangle\Phi}(\Gamma, M)$ *denotes the homology group of* $Y^\triangle \otimes_\Phi M$ *with* $Y^\triangle$ *the right* Φ-*resolution modified as in* (1.1) *of a right* Φ-*projective resolution* Y *of* Γ. □

First we will define a *restriction map* $\mathrm{Res}^r : H^r(\Gamma, M) \to H^r(\Lambda, M)$ for $r \in \mathbb{Z}$. Let $\mathrm{res}^p : H^p(\Gamma, M) \to \mathrm{Ext}^p_\Phi(\Gamma, M)$ for $p \geq 1$ be the homomorphism induced from a natural map $\mathrm{Hom}_\Sigma((X_\Gamma)_p, M) \to \mathrm{Hom}_\Phi((X_\Gamma)_p, M)$, and let $\mathrm{res}_q : H_q^\triangle(\Gamma, M) \to$

$\mathrm{Tor}_q^{\triangle\Phi}(\Gamma, M)$ for $q \geq 1$ be the homomorphism induced from

$$(X_\Gamma)_q^\triangle \otimes_\Sigma M \to (X_\Gamma)_q^\triangle \otimes_\Phi M\,, \qquad y \otimes_\Sigma x \mapsto \sum_{i=1}^m ya_i \otimes_\Phi b_i x\,.$$

Then, composing the isomorphisms in Lemma 1.1, it will be seen that

$$\widetilde{\mathrm{res}}^p : H^p(\Gamma, M) \to H^p(\Lambda, M)$$

for $p \geq 1$ is induced from a natural map $\mathrm{Hom}_\Sigma((X_\Gamma)_p, M) \to \mathrm{Hom}_\Omega((X_\Lambda)_p, M)$, and

$$\widetilde{\mathrm{res}}_q : H_q^\triangle(\Gamma, M) \to H_q^{\triangle'}(\Lambda, M)$$

for $q \geq 1$ is induced from

$$(X_\Gamma)_q^\triangle \otimes_\Sigma M \to (X_\Lambda)_q^{\triangle'} \otimes_\Omega M\,,$$

$$[y_1, \ldots, y_q] \otimes_\Sigma x \mapsto \sum_{i_1, \ldots, i_{q+1}=1}^m [\mu_{\Gamma/\Lambda}(b_{i_1} y_1 a_{i_2}), \ldots, \mu_{\Gamma/\Lambda}(b_{i_q} y_q a_{i_{q+1}})] \otimes_\Omega b_{i_{q+1}} x a_{i_1}^\triangle\,.$$

Furthermore we set

$$\widetilde{\mathrm{res}}^0 = \iota_M\,, \qquad \widetilde{\mathrm{res}}_0(x) = \sum_{i=1}^m b_i x a_i^\triangle \quad (x \in M)\,.$$

Then we have the following commutative diagram:

$$\begin{tikzcd}
\cdots & \mathrm{Hom}_\Sigma((X_\Gamma)_1, M) \arrow[l] \arrow[d, "\widetilde{\mathrm{res}}^1"] & M \arrow[l, "d_1^\#"'] \arrow[d, "\widetilde{\mathrm{res}}^0"] & M \arrow[l, "N_\Gamma"'] \arrow[d, "\widetilde{\mathrm{res}}_0"] & (X_\Gamma)_1^\triangle \otimes_\Sigma M \arrow[l, "d_1 \otimes \iota"'] \arrow[d, "\widetilde{\mathrm{res}}_1"] & \cdots \arrow[l] \\
\cdots & \mathrm{Hom}_\Omega((X_\Lambda)_1, M) \arrow[l] & M \arrow[l, "d_1^\#"'] & M \arrow[l, "N_\Lambda"'] & (X_\Lambda)_1^{\triangle'} \otimes_\Omega M \arrow[l, "d_1 \otimes \iota"'] & \cdots \arrow[l]
\end{tikzcd}$$

Thus we can define the restriction map

$$\mathrm{Res}^r : H^r(\Gamma, M) \to H^r(\Lambda, M) \quad (r \in \mathbb{Z})$$

such that $\mathrm{Res}^r = \widetilde{\mathrm{res}}^r$ $(r \geq 0)$, $= \widetilde{\mathrm{res}}_{-r-1}$ $(r \leq -1)$. In particular, we have

$$\mathrm{Res}^0 : M^\Gamma / N_\Gamma(M) \to M^\Lambda / N_\Lambda(M)\,, \qquad \bar{x} \mapsto \bar{x}\,,$$

$$\mathrm{Res}^{-1} : {}_{N_\Gamma}M / I_\Gamma(M) \to {}_{N_\Lambda}M / I_\Lambda(M)\,, \qquad \bar{x} \mapsto \overline{\sum_{i=1}^m b_i x a_i^\triangle}$$

(see [7, Section 1] for notation).

Next we will define a *corestriction map* $\mathrm{Cor}^r : H^r(\Lambda, M) \to H^r(\Gamma, M)$ $(r \in \mathbb{Z})$. Let $\mathrm{cor}^p : \mathrm{Ext}^p_\Phi(\Gamma, M) \to H^p(\Gamma, M)$ for $p \geq 1$ be the homomorphism induced from

$$\mathrm{Hom}_\Phi((X_\Gamma)_p, M) \to \mathrm{Hom}_\Sigma((X_\Gamma)_p, M)\,,$$

$$f \mapsto \Big(x \mapsto \sum_i a_i f(b_i x)\Big)\,,$$

and let $\mathrm{cor}_q : \mathrm{Tor}^{\triangle\Phi}_q(\Gamma, M) \to H^\triangle_q(\Gamma, M)$ for $q \geq 1$ be the homomorphism induced from a natural map $(X_\Gamma)^\triangle_q \otimes_\Phi M \to (X_\Gamma)^\triangle_q \otimes_\Sigma M$. Then, composing the isomorphisms in Lemma 1.1, it will be seen that

$$\widetilde{\mathrm{cor}}^p : H^p(\Lambda, M) \to H^p(\Gamma, M)$$

for $p \geq 1$ is induced from

$$\mathrm{Hom}_\Omega((X_\Lambda)_p, M) \to \mathrm{Hom}_\Sigma((X_\Gamma)_p, M)\,,$$

$$g \mapsto \Big(y_0[y_1, \ldots, y_p]y_{p+1} \mapsto \sum_{i_1, \ldots, i_{p+1}=1}^{m} y_0 a_{i_1} \cdot g([\mu_{\Gamma/\Lambda}(b_{i_1} y_1 a_{i_2}), \ldots, \mu_{\Gamma/\Lambda}(b_{i_p} y_p a_{i_{p+1}})]) \cdot b_{i_{p+1}} y_{p+1}\Big)\,,$$

and

$$\widetilde{\mathrm{cor}}_q : H^{\triangle'}_q(\Lambda, M) \to H^\triangle_q(\Gamma, M)$$

for $q \geq 1$ is induced from a natural map $(X_\Lambda)^{\triangle'}_q \otimes_\Omega M \to (X_\Gamma)^\triangle_q \otimes_\Sigma M$. Furthermore we set

$$\widetilde{\mathrm{cor}}^0 = N_{\Gamma/\Lambda}\,, \qquad \widetilde{\mathrm{cor}}_0 = \iota_M\,.$$

Then we have the following commutative diagram:

$$\begin{tikzcd}
\cdots & \mathrm{Hom}_\Sigma((X_\Gamma)_1, M) \arrow[l] & M \arrow[l, "d_1^\#"'] & M \arrow[l, "N_\Gamma"'] & (X_\Gamma)^\triangle_1 \otimes_\Sigma M \arrow[l, "d_1 \otimes \iota"'] & \cdots \arrow[l] \\
\cdots & \mathrm{Hom}_\Omega((X_\Lambda)_1, M) \arrow[l] \arrow[u, "\widetilde{\mathrm{cor}}^1"'] & M \arrow[l, "d_1^\#"'] \arrow[u, "\widetilde{\mathrm{cor}}^0"'] & M \arrow[l, "N'_\Lambda"'] \arrow[u, "\widetilde{\mathrm{cor}}_0"'] & (X_\Lambda)^{\triangle'}_1 \otimes_\Omega M \arrow[l, "d_1 \otimes \iota"'] \arrow[u, "\widetilde{\mathrm{cor}}_1"'] & \cdots \arrow[l]
\end{tikzcd}$$

Thus we can define the corestriction map

$$\mathrm{Cor}^r : H^r(\Lambda, M) \to H^r(\Gamma, M) \quad (r \in \mathbb{Z})$$

such that $\mathrm{Cor}^r = \widetilde{\mathrm{cor}}^r$ $(r \geq 0)$, $= \widetilde{\mathrm{cor}}_{-r-1}$ $(r \leq -1)$. In particular, we have

$$\mathrm{Cor}^0 : M^{\Lambda}/N_{\Lambda}(M) \to M^{\Gamma}/N_{\Gamma}(M)\,, \qquad \bar{x} \mapsto \overline{N_{\Gamma/\Lambda}(x)}\,,$$
$$\mathrm{Cor}^{-1} : {}_{N_{\Lambda}}M/I_{\Lambda}(M) \to {}_{N_{\Gamma}}M/I_{\Gamma}(M)\,, \qquad \bar{x} \mapsto \bar{x}\,.$$

These maps satisfy the following fundamental properties:

Lemma 1.2. (i) *Given homomorphism $f : A \to B$ of left Σ-modules,*

$$f^* \cdot \mathrm{Res}^r = \mathrm{Res}^r \cdot f^* : H^r(\Gamma, A) \to H^r(\Lambda, B) \quad (r \in \mathbb{Z})\,,$$
$$f^* \cdot \mathrm{Cor}^r = \mathrm{Cor}^r \cdot f^* : H^r(\Lambda, A) \to H^r(\Gamma, B) \quad (r \in \mathbb{Z})\,,$$

where f^* denotes the appropriate homomorphism of cohomology groups induced from f.

(ii) *Given short exact sequence $0 \to A \to B \to C \to 0$,*

$$\partial \cdot \mathrm{Res}^r = \mathrm{Res}^{r+1} \cdot \partial : H^r(\Gamma, C) \to H^{r+1}(\Lambda, A) \quad (r \in \mathbb{Z})\,,$$
$$\partial \cdot \mathrm{Cor}^r = \mathrm{Cor}^{r+1} \cdot \partial : H^r(\Lambda, C) \to H^{r+1}(\Gamma, A) \quad (r \in \mathbb{Z})\,,$$

where ∂ denotes the appropriate connecting homomorphism.

(iii) *Given left Σ-module M,*

$$\mathrm{Cor}^r \cdot \mathrm{Res}^r(w) = N_{\Gamma/\Lambda}(1)w \quad (r \in \mathbb{Z})$$

for $w \in H^r(\Gamma, M)$.

Proof. (i) It is easily verified. (ii) follows from the fact that $(X_\Gamma)_p$ and $(X_\Lambda)_p$ $(p \geq 0)$ are left Σ-projective and left Ω-projective, respectively, and that $(X_\Gamma)_q^{\triangle}$and $(X_\Lambda)_q^{\triangle'}$ $(q \geq 0)$ are right Σ-flat and right Ω-flat, respectively (cf. [7, Lemma 1.1]). (iii) Note that $N_{\Gamma/\Lambda}(1) \in Z\Gamma$ and the $Z\Gamma$-operation on $H^r(\Gamma, M)$ (cf. [7, Section 1]). Then the statement will follow from the constructions of res^p, res_q, cor^p, cor_q, etc. □

Remark 2. The restriction map $\mathrm{Res}^r : H^r(\Gamma, M) \to H^r(\Lambda, M)$ $(r \in \mathbb{Z})$ may be defined to be the map induced from a natural homomorphism $\mathrm{Hom}_\Sigma((X_\Gamma)_r, M) \to \mathrm{Hom}_\Omega((X_\Gamma)_r, M)$ $(r \in \mathbb{Z})$, by means of a complete resolution $(X_\Gamma)_r$ $(r \in \mathbb{Z})$ (cf. [7, Section 1]). And also the corestriction map $\mathrm{Cor}^r : H^{\triangle'}_{-r-1}(\Lambda, M) \to H^{\triangle}_{-r-1}(\Gamma, M)$ $(r \in \mathbb{Z})$ may be defined to be the map induced from a natural homomorphism $(X_\Gamma)^{\triangle'}_{-r-1} \otimes_\Omega M \to (X_\Gamma)^{\triangle}_{-r-1} \otimes_\Sigma M$. But these definitions are inconvenient for explicit calculation of these maps and to see the relationship between them. Accordingly, as in the above, we defined Res^r and Cor^r on positive degree and negative degree separately, and then combine them. Of course, these two kind of definitions coincide, which will be seen by *dimension-shifting* [7, Corollary 1.5] and Lemma 1.2.

Given left Σ-modules A and B, let $A \otimes_\Lambda B \to A \otimes_\Gamma B$ be a canonical left Σ-module homomorphism induced from the embedding $\iota : \Lambda \hookrightarrow \Gamma$. Then we have a *modified cup product*

$$\cup_\iota : H^r(\Lambda, A) \otimes_{Z\Lambda} H^s(\Lambda, B) \to H^{r+s}(\Lambda, A \otimes_\Gamma B)$$

(see [7, Section 2]).

Lemma 1.3. *The following diagram commutes*:

$$\begin{array}{ccc} H^r(\Gamma, A) \otimes_R H^s(\Gamma, B) & \xrightarrow{\mathrm{Res}^r \otimes \mathrm{Res}^s} & H^r(\Lambda, A) \otimes_{Z\Lambda} H^s(\Lambda, B) \\ \downarrow{\scriptstyle \cup} & & \downarrow{\scriptstyle \cup_\iota} \\ H^{r+s}(\Gamma, A \otimes_\Gamma B) & \xrightarrow{\mathrm{Res}^{r+s}} & H^{r+s}(\Lambda, A \otimes_\Gamma B) \end{array} \tag{1.4}$$

Proof. We use induction on a pair of integers (r, s). The result is clear from the definition of cup product [7, Section 2, (PIII)] when $(r, s) = (0, 0)$. Here, we notice that the modified cup product $\cup_\iota$ satisfies the similar condition as [7, Section 2 (PII)]. Assume that the diagram (1.4) is commutative for some (r, s). Then, by means of four kinds of *dimension-shifting* [7, Corollary 1.5] and Lemma 1.2(ii), we know that the diagram (1.4) is commutative for $(r \pm 1, s \pm 1)$ (cf. [7, Section 2.2]). $\square$

Thus Res gives a ring homomorphism of graded rings:

$$\mathrm{Res}^* : (H^*(\Gamma, \Gamma), \cup) \to (H^*(\Lambda, \Gamma), \cup_\iota)\,.$$

2. Complete cohomology of crossed products

2.1. Conjugation

Let R be a commutative ring and Λ a Frobenius commutative R-algebra of free rank n. We suppose that Λ has a finite group G of order m of automorphisms over R and $\Lambda^G = R$. We assume an additional condition:

(C) For every $\sigma\,(\neq 1) \in G$, there exists $x \in \Lambda$ such that $\sigma(x) - x$ is not a zero-divisor.

Notation and assumption being as above, we consider a crossed product $\Gamma = (\Lambda, G, \theta) = \sum_{\sigma \in G} \bigoplus \Lambda w_\sigma$ $(w_\sigma w_\tau = \theta(\sigma, \tau) w_{\sigma\tau})$ for a normalized factor set θ with its values in the group $U(\Lambda)$ of all units in Λ. The condition (C) implies $Z\Gamma = R$. It is easy to see that the extension Γ/Λ is a Frobenius extension of free rank m with the dual bases $(w_\sigma), (w_\sigma^{-1})$ $(\sigma \in G)$. Accordingly, we will succeed the notation as in Section 1 and we notice that $\{a_i\} = \{w_\sigma\}$, $\{b_i\} = \{w_\sigma^{-1}\}$,

$\mu_{\Gamma/\Lambda}(w_\tau^{-1}w_\sigma)=\delta_{\sigma,\tau}$, $w_\tau^{\triangle}=w_\tau(\sum_{i=1}^n u_i^{\tau^{-1}}\mu_\Lambda(v_i^{\tau^{-1}}))$, ${}^{\triangle'}=\iota_\Lambda$ and $\Gamma^\Lambda=\Lambda$, where we put $y^\tau=w_\tau y w_\tau^{-1}$ for $y\in\Lambda$ and $\tau\in G$.

In this section, we will define a *conjugation map* $(\gamma_\tau)^r: H^r(\Lambda, M)\to H^r(\Lambda, M)$ ($r\in\mathbb{Z}$) for $\tau\in G$ and a left Σ-module M dividing into four cases, as in Section 1; $r>0$, $r=0$, $r=-1$ and $r<-1$.

First let

$$(\gamma_\tau')^p: \mathrm{Ext}_\Phi^p(\Gamma, M)\to \mathrm{Ext}_\Phi^p(\Gamma, M)\quad (p\geq 1)$$

be the map induced from

$$\begin{aligned}&\mathrm{Hom}_\Phi((X_\Gamma)_p, M)\to \mathrm{Hom}_\Phi((X_\Gamma)_p, M)\,,\\&g\mapsto (x\mapsto w_\tau\cdot g(w_\tau^{-1}x))\,,\end{aligned}$$

and let

$$(\gamma_\tau')_q: \mathrm{Tor}_q^{\triangle\Phi}(\Gamma, M)\to \mathrm{Tor}_q^{\triangle\Phi}(\Gamma, M)\quad (q\geq 1)$$

be the map induced from

$$\begin{aligned}&(X_\Gamma)_q^\triangle\otimes_\Phi M\to (X_\Gamma)_q^\triangle\otimes_\Phi M\,,\\&x\otimes_\Phi y\mapsto xw_\tau^{-1}\otimes_\Phi w_\tau y\,.\end{aligned}$$

Composing the isomorphisms as in Lemma 1.1, we have the following homomorphisms:

$$\begin{aligned}&(\tilde\gamma_\tau)^p: H^p(\Lambda, M)\to H^p(\Lambda, M)\,,\quad \mathrm{Hom}_\Omega((X_\Lambda)_p, M)\to \mathrm{Hom}_\Omega((X_\Lambda)_p, M)\,,\\&g\mapsto (y_0[y_1,\dots,y_p]y_{p+1}\mapsto w_\tau\cdot g(y_0^{\tau^{-1}}[y_1^{\tau^{-1}},\dots,y_p^{\tau^{-1}}]y_{p+1}^{\tau^{-1}})\cdot w_\tau^{-1})\end{aligned}$$

and

$$\begin{aligned}&(\tilde\gamma_\tau)_q: H_q(\Lambda, M)\to H_q(\Lambda, M)\,,\quad (X_\Lambda)_q\otimes_\Omega M\to (X_\Lambda)_q\otimes_\Omega M\,,\\&[y_1,\dots,y_p]\otimes_\Omega a\mapsto [y_1^\tau,\dots,y_p^\tau]\otimes_\Omega w_{\tau^{-1}}^{-1}aw_{\tau^{-1}}^\triangle\,.\end{aligned}\tag{2.1}$$

Furthermore we set

$$\begin{aligned}&(\tilde\gamma_\tau)^0: M\to M\,,\qquad a\mapsto w_\tau a w_\tau^{-1}\,,\\&(\tilde\gamma_\tau)_0: M\to M\,,\qquad a\mapsto w_{\tau^{-1}}^{-1}aw_{\tau^{-1}}^\triangle\,.\end{aligned}\tag{2.2}$$

Then the following diagram commutes:

$$\begin{array}{ccccccc}
\cdots \longleftarrow \mathrm{Hom}_\Omega((X_\Lambda)_1, M) & \xleftarrow{d_1^\#} & M & \xleftarrow{N_\Lambda} & M & \xleftarrow{d_1\otimes\iota} & (X_\Lambda)_1 \otimes_\Omega M \longleftarrow \cdots \\
\downarrow (\tilde{\gamma}_\tau)^1 & & \downarrow (\tilde{\gamma}_\tau)^0 & & \downarrow (\tilde{\gamma}_\tau)_0 & & \downarrow (\tilde{\gamma}_\tau)_1 \\
\cdots \longleftarrow \mathrm{Hom}_\Omega((X_\Lambda)_1, M) & \xleftarrow{d_1^\#} & M & \xleftarrow{N_\Lambda} & M & \xleftarrow{d_1\otimes\iota} & (X_\Lambda)_1 \otimes_\Omega M \longleftarrow \cdots
\end{array}$$

Thus we can define the conjugation map

$$(\gamma_\tau)^r : H^r(\Lambda, M) \to H^r(\Lambda, M) \quad (r \in \mathbb{Z})$$

such that $(\gamma_\tau)^r = (\tilde{\gamma}_\tau)^r$ $(r \geq 0)$, $=(\tilde{\gamma}_\tau)_{-r-1}$ $(r \leq -1)$. Then it will be seen that γ_τ satisfies the following fundamental properties:

Lemma 2.1. (i) *Given homomorphism $f : A \to B$ of left Σ-modules,*

$$\gamma_\tau \cdot f^* = f^* \cdot \gamma_\tau : H^r(\Lambda, A) \to H^r(\Lambda, B)\,,$$

where f^* denotes the appropriate homomorphism of cohomology groups induced from f.

(ii) *Given short exact sequence $0 \to A \to B \to C \to 0$,*

$$\gamma_\tau \cdot \partial = \partial \cdot \gamma_\tau : H^r(\Lambda, C) \to H^{r+1}(\Lambda, A)\,,$$

where ∂ denotes the appropriate connecting homomorphism.

(iii) $\qquad \gamma_\sigma \cdot \gamma_\tau = \gamma_{\sigma\tau}\,, \qquad \gamma_1 = \iota\,.$

Hence γ_τ is an isomorphism. □

Lemma 2.2. *The following diagram commutes*:

$$\begin{array}{ccc}
H^r(\Lambda, A) \otimes_R H^s(\Lambda, B) & \xrightarrow{(\gamma_\tau)^r \otimes (\gamma_\tau)^s} & H^r(\Lambda, A) \otimes_R H^s(\Lambda, B) \\
\downarrow \cup_\iota & & \downarrow \cup_\iota \\
H^{r+s}(\Lambda, A \otimes_\Gamma B) & \xrightarrow{(\gamma_\tau)^{r+s}} & H^{r+s}(\Lambda, A \otimes_\Gamma B)
\end{array}$$

Proof. Similar to the proof of Lemma 1.3. □

Therefore, γ_τ is an automorphism on the graded ring $(H^*(\Lambda, \Gamma), \cup_\iota)$.

2.2. Main results

We use the same notation as in Section 2.1 with $R = \mathbb{Z}$. Then $N_\Gamma(\Gamma)$ is an ideal of $\mathbb{Z}$. Hence we denote its generator of the ideal by $N_\Gamma(\Gamma)$ again; $N_\Gamma(\Gamma) \in \mathbb{Z}$. Since

$$N_\Gamma(\Gamma) \cdot H^r(\Gamma, M) = 0 \quad (r \in \mathbb{Z})$$

for any left Σ-module M (cf. [1, Section 4]), we have a primary decomposition:

$$H^r(\Gamma, M) = \bigoplus_{l|N_\Gamma(\Gamma)} H^r(\Gamma, M)_{(l)} . \tag{2.3}$$

For a prime integer l such that $l \mid N_\Gamma(\Gamma)$ and $l \nmid |G|$,

$$\mathrm{Cor}^r \cdot \mathrm{Res}^r = N_{\Gamma/\Lambda}(1) = |G| : H^r(\Gamma, M)_{(l)} \to H^r(\Gamma, M)_{(l)}$$

is an isomorphism. It follows that Res^r is a monomorphism and Cor^r is an epimorphism on the l-primary component. We set

$$[H^r(\Lambda, M)]^G = \{x \in H^r(\Lambda, M) \mid (\gamma_\tau)^r(x) = x \text{ for all } \tau \in G\} .$$

Then we have the following proposition:

Proposition 2.3. *For a prime integer l such that $l \mid N_\Gamma(\Gamma)$ and $l \nmid |G|$,*

$$\mathrm{Res}^r(H^r(\Gamma, M)_{(l)}) = [H^r(\Lambda, M)_{(l)}]^G \tag{2.4}$$

holds.

Proof. It suffices to show that

(i) $$\mathrm{res}^p(H^p(\Gamma, M)_{(l)}) = [\mathrm{Ext}^p_\Phi(\Gamma, M)_{(l)}]^G$$
$$:= \{\alpha \in \mathrm{Ext}^p_\Phi(\Gamma, M)_{(l)} \mid \gamma'_\tau(\alpha) = \alpha \text{ for all } \tau \in G\}$$

for $p \geq 1$,

(ii) $$\mathrm{res}_q(H^\triangle_q(\Gamma, M)_{(l)}) = [\mathrm{Tor}^{\triangle\Phi}_q(\Gamma, M)_{(l)}]^G$$
$$:= \{\beta \in \mathrm{Tor}^{\triangle\Phi}_q(\Gamma, M)_{(l)} \mid \gamma'_\tau(\beta) = \beta \text{ for all } \tau \in G\}$$

for $q \geq 1$,

and (iii) (2.4) holds for $r = 0$ and for $r = 1$. In the following, we will prove them.

(i) For $[g] \in H^p(\Gamma, M)_{(l)}$ ($g \in \mathrm{Hom}_\Sigma((X_\Gamma)_p, M)$), we have $\gamma'_\tau(\mathrm{res}^p([g])) = [(x \mapsto w_\tau \cdot g(w_\tau^{-1}x))] = [g] = \mathrm{res}^p([g])$. Conversely, let $[h] \in [\mathrm{Ext}^p_\Phi(\Gamma, M)_{(l)}]^G$. Then we have

$$\mathrm{res}^p \cdot \mathrm{cor}^p([h]) = \mathrm{res}^p\left(\left[\left(x \mapsto \sum_{\sigma \in G} w_\sigma h(w_\sigma^{-1}x)\right)\right]\right) = |G|\,[h] .$$

Since $l \nmid |G|$, it follows that $\mathrm{res}^p(|G|^{-1}\,\mathrm{cor}^p([h])) = [h]$. Thus (i) was shown.

(ii) For $[y \otimes_\Sigma x] \in H^\triangle_q(\Gamma, M)_{(l)}$ ($y \otimes_\Sigma x \in (X_\Gamma)^\triangle_q \otimes_\Sigma M$), we have

$$\begin{aligned}
&\gamma'_\tau(\mathrm{res}_q([y\otimes_\Sigma x]))\\
&\quad=\gamma'_\tau\left(\left[\sum_\sigma yw_\sigma\otimes_\Phi w_\sigma^{-1}x\right]\right)\\
&\quad=\left[\sum_\sigma yw_\sigma w_\tau^{-1}\otimes_\Phi w_\tau w_\sigma^{-1}x\right]\\
&\quad=\left[\sum_\sigma yw_\sigma\otimes_\Phi w_\sigma^{-1}x\right]\\
&\quad=\mathrm{res}_q([y\otimes_\Sigma x])\,.
\end{aligned}$$

Conversely, let $[y\otimes_\Phi x]\in[\mathrm{Tor}_q^{\triangle\Phi}(\Gamma, M)_{(l)}]^G$. Then we have

$$\begin{aligned}
&\mathrm{res}_q\cdot\mathrm{cor}_q([y\otimes_\Phi x])\\
&\quad=\mathrm{res}_q([y\otimes_\Sigma x])\\
&\quad=\left[\sum_\sigma yw_\sigma\otimes_\Phi w_\sigma^{-1}x\right]\\
&\quad=\sum_\sigma \gamma_{\sigma^{-1}}([y\otimes_\Phi x])\\
&\quad=|G|\,[y\otimes_\Phi x]\,.
\end{aligned}$$

Thus (ii) was shown.

(iii) In case of $r=0$: For $\bar{x}\in H^0(\Gamma, M)_{(l)}=(M^\Gamma/N_\Gamma(M))_{(l)}$ $(x\in M^\Gamma)$, we have

$$\gamma_\tau(\mathrm{Res}^0(\bar{x}))=\overline{w_\tau x w_\tau^{-1}}=\bar{x}=\mathrm{Res}^0(\bar{x})\,.$$

Conversely, let $\bar{x}\in[H^0(\Lambda, M)_{(l)}]^G=[(M^\Lambda/N_\Lambda(M))_{(l)}]^G$ $(x\in M^\Lambda)$. Then we have

$$\begin{aligned}
&\mathrm{Res}^0\cdot\mathrm{Cor}^0(\bar{x})\\
&\quad=\mathrm{Res}^0\left(\overline{\sum_\sigma w_\sigma x w_\sigma^{-1}}\right)\\
&\quad=\overline{\sum_\sigma w_\sigma x w_\sigma^{-1}}\\
&\quad=\sum_\sigma \gamma_\sigma(\bar{x})\\
&\quad=|G|\,\bar{x}\,.
\end{aligned}$$

In case of $r=-1$: For $\bar{x}\in H^{-1}(\Gamma, M)_{(l)}=({}_{N_\Gamma}M/I_\Gamma(M))_{(l)}$ $(x\in {}_{N_\Gamma}M)$, we have

$$\begin{aligned}
&\gamma_\tau(\mathrm{Res}^{-1}(\bar{x}))\\
&= \gamma_\tau\left(\overline{\sum_\sigma w_\sigma^{-1} x w_\sigma^{\triangle}}\right)\\
&= \overline{\sum_\sigma w_{\tau^{-1}}^{-1} w_\sigma^{-1} x w_\sigma^{\triangle} w_{\tau^{-1}}^{\triangle}}\\
&= \overline{\sum_\sigma (\theta(\sigma,\tau^{-1})^{\tau\sigma^{-1}})^{-1} w_{\sigma\tau^{-1}}^{-1} x w_{\sigma\tau^{-1}}^{\triangle} \theta(\sigma,\tau^{-1})^{\tau\sigma^{-1}}}\\
&= \overline{\sum_\sigma w_{\sigma\tau^{-1}}^{-1} x w_{\sigma\tau^{-1}}^{\triangle}}\\
&= \mathrm{Res}^{-1}(\bar{x})\,.
\end{aligned}$$

Conversely, let $\bar{x} \in [H^{-1}(\Lambda, M)_{(l)}]^G = [({}_{N_\Lambda}M/I_\Lambda(M))_{(l)}]^G$ $(x \in {}_{N_\Lambda}M)$. Then we have

$$\begin{aligned}
&\mathrm{Res}^{-1}\cdot\mathrm{Cor}^{-1}(\bar{x})\\
&= \mathrm{Res}^{-1}(\bar{x})\\
&= \overline{\sum_\sigma w_\sigma^{-1} x w_\sigma^{\triangle}}\\
&= \sum_\sigma \gamma_{\sigma^{-1}}(\bar{x})\\
&= |G|\,\bar{x}\,.
\end{aligned}$$

This completes the proof of the proposition. □

Theorem 2.4. *Let l be a prime integer such that $l \mid N_\Gamma(\Gamma)$ and $l \nmid |G|$. Assume that the group of unit elements in $(\Lambda/N_\Lambda(\Gamma))_{(l)}$ is a finite group. Then the graded ring $H^*(\Lambda,\Gamma)_{(l)} = \sum_{r\in\mathbb{Z}} \bigoplus H^r(\Lambda,\Gamma)_{(l)}$ has an invertible element of nonzero degree if and only if so does $H^*(\Gamma,\Gamma)_{(l)} = \sum_{r\in\mathbb{Z}} \bigoplus H^r(\Gamma,\Gamma)_{(l)}$. Accordingly, in such a case, the l-primary component of the cohomology group of Γ is periodic.*

Proof. The 'if' part is obtained from the fact that there is a ring homomorphism $\mathrm{Res}^* : H^*(\Gamma,\Gamma)_{(l)} \to H^*(\Lambda,\Gamma)_{(l)}$. Let $u \in H^d(\Lambda,\Gamma)_{(l)}$ for $d \neq 0$ be an invertible element in $H^*(\Lambda,\Gamma)_{(l)}$. By Lemma 2.2, $\gamma_\tau(u) \in H^d(\Lambda,\Gamma)_{(l)}$ is also invertible in $H^*(\Lambda,\Gamma)_{(l)}$ for any $\tau \in G$. Then u and $\gamma_\tau(u)$ derive the isomorphisms of Λ-modules, respectively:

$$H^d(\Lambda,\Gamma)_{(l)} \xrightarrow{\sim} H^0(\Lambda,\Gamma)_{(l)} = (\Lambda/N_\Lambda(\Gamma))_{(l)}\,,$$

$$z \mapsto z \cup_\iota u^{-1}\,, \qquad z \mapsto z \cup_\iota \gamma_\tau(u)^{-1}\,.$$

Hence $\gamma_\tau(u) = \lambda u$ for a unit element $\bar{\lambda} \in (\Lambda/N_\Lambda(\Gamma))_{(l)}$ $(\lambda \in \Lambda)$. Note that λ depends on τ. However, there exists a common integer $e > 0$ for all $\tau \in G$ such that $\gamma_\tau(u^e) = \lambda^e u^e = u^e$, where we put $u^e = u \cup_\iota \cdots \cup_\iota u$ (e times), $\cup_\iota$ denoting the modified cup product on $H^*(\Lambda, \Gamma)$ ($\iota : \Lambda \hookrightarrow \Gamma$). This says $u^e \in [H^{de}(\Lambda, \Gamma)_{(l)}]^G$. Of course, u^e is still invertible. Since there is a ring isomorphism:

$$\mathrm{Res}^* : H^*(\Gamma, \Gamma)_{(l)} \xrightarrow{\sim} [H^*(\Lambda, \Gamma)_{(l)}]^G$$

by Lemma 1.3 and Proposition 2.3, it follows that $H^{de}(\Gamma, \Gamma)_{(l)}$ ($de \neq 0$) has an invertible element in $H^*(\Gamma, \Gamma)_{(l)}$. The last statement follows from [7, Lemma 3.1]. □

3. Applications

Let m be a square-free integer. Λ denotes the ring of integers in the quadratic field $\mathbb{Q}(\sqrt{m})$. We denote the Galois group of the extension $\mathbb{Q}(\sqrt{m})/\mathbb{Q}$ by $G = \langle \sigma \rangle$, $\sigma(\sqrt{m}) = -\sqrt{m}$.

First, we treat the case of $m \equiv 2,3 \pmod 4$. $\Lambda = \mathbb{Z}[\sqrt{m}]$ is a Frobenius $\mathbb{Z}$-algebra with $(1, \sqrt{m})$, $(\sqrt{m}, 1)$ as the $\mathbb{Z}$-dual bases. We set $\Gamma = \Lambda \oplus \Lambda w_\sigma$, $w_\sigma^2 = \pm 1$; then $N_\Gamma(\Gamma) = 4m$. Note that $w_\sigma^\triangle = -w_\sigma$.

Proposition 3.1. *Notation and assumption being as above, let M be a Σ-module and let l be a prime integer such that $l \mid m$ and $l \neq 2$. Then $H^{-2}(\Gamma, \Gamma)_{(l)}$ has an invertible element in $H^*(\Gamma, \Gamma)_{(l)}$, and so $H^r(\Gamma, M)_{(l)}$ is periodic of period 2. In particular, $H^r(\Gamma, \Gamma)_{(l)} \simeq \mathbb{Z}/l\mathbb{Z}$ for r even, $=0$ for r odd.*

Proof. Since $H^{-2}(\Lambda, \Lambda w_\sigma) \simeq H^0(\Lambda, \Lambda w_\sigma) = 0$, we have $H^{-2}(\Lambda, \Gamma) \simeq H^{-2}(\Lambda, \Lambda)$. By means of an isomorphism

$$\partial : H^{-2}(\Lambda, \Lambda) \xrightarrow{\sim} H^{-1}(\Lambda, K(\Lambda)) =_{N_\Lambda} K(\Lambda)/I_\Lambda(K(\Lambda))$$

obtained from a short exact sequence $0 \to K(\Lambda) \to \Lambda \otimes \Lambda \to \Lambda \to 0$, we have the following isomorphism (see [7, Proposition 3.8]):

$$\begin{aligned} H^{-2}(\Lambda, \Gamma) \simeq H^{-2}(\Lambda, \Lambda) &\xrightarrow[\partial]{\sim} H^{-1}(\Lambda, K(\Lambda)) =_{N_\Lambda} K(\Lambda)/I_\Lambda(K(\Lambda)) \\ &= (\Lambda/N_\Lambda(\Lambda))(1 \otimes \sqrt{m} - \sqrt{m} \otimes 1) \xrightarrow{\sim} \Lambda/N_\Lambda(\Lambda) . \end{aligned}$$

This implies that $H^{-2}(\Lambda, \Gamma)$ has an invertible element u associated to $1 \otimes \sqrt{m} - \sqrt{m} \otimes 1$. Considering the above isomorphisms on the chain level, we have

$$\begin{aligned} &(X_\Lambda)_1 \otimes_\Omega \Gamma \leftarrow (X_\Lambda)_1 \otimes_\Omega \Lambda \xrightarrow{\partial} (X_\Lambda)_0 \otimes_\Omega K(\Lambda) = K(\Lambda) , \\ &[y] \otimes_\Omega x \leftarrow\!\shortmid [y] \otimes_\Omega x \mapsto 1 \otimes xy - y \otimes x . \end{aligned}$$

Since $\tilde{\gamma}_\sigma([y]\otimes_\Omega x) = -[y^\sigma]\otimes_\Omega x^\sigma$ by (2.1), it follows that $\tilde{\gamma}_\sigma(u) = u$. Therefore, by the proof of Theorem 2.4, $H^{-2}(\Gamma, \Gamma)_{(l)}$ has an invertible element in $H^*(\Gamma, \Gamma)_{(l)}$, that is, $H^r(\Gamma, M)_{(l)}$ is periodic of period 2.

The statement in case of r = even immediately follows from

$$H^0(\Gamma, \Gamma)_{(l)} = (\mathbb{Z}/N_\Gamma(\Gamma))_{(l)} = (\mathbb{Z}/4m\mathbb{Z})_{(l)} \simeq \mathbb{Z}/l\mathbb{Z} .$$

Now we have the following isomorphisms:

$$\begin{aligned} &H^{-1}(\Lambda, \Gamma)_{(l)} \\ &\quad = ({}_{N_\Lambda}\Gamma/I_\Lambda(\Gamma))_{(l)} \\ &\quad = (\Lambda w_\sigma/2\sqrt{m}(\Lambda w_\sigma))_{(l)} \\ &\quad = \{(\mathbb{Z} + \mathbb{Z}\sqrt{m})w_\sigma/(2m\mathbb{Z} + 2\mathbb{Z}\sqrt{m})w_\sigma\}_{(l)} \\ &\quad \simeq \mathbb{Z}/l\mathbb{Z} . \end{aligned}$$

By (2.2), we have $\gamma_\sigma((a + b\sqrt{m})w_\sigma) = w_\sigma^{-1}((a + b\sqrt{m})w_\sigma)w_\sigma^\triangle = -(a - b\sqrt{m})w_\sigma$. This means $\gamma_\sigma = -\iota$ on $H^{-1}(\Lambda, \Gamma)_{(l)}$, hence, by Proposition 2.3, $H^{-1}(\Gamma, \Gamma)_{(l)} \simeq [H^{-1}(\Lambda, \Gamma)_{(l)}]^G = 0$. □

Remark 3. By direct calculations, we have

$$\begin{aligned} &H^1(\Gamma, \Gamma) \simeq (\mathbb{Z}/2\mathbb{Z})^{(3)} \quad \text{(3-times direct sum and so on)} , \\ &H^0(\Gamma, \Gamma) \simeq \mathbb{Z}/4m\mathbb{Z}, \\ &H^{-1}(\Gamma, \Gamma) \simeq (\mathbb{Z}/2\mathbb{Z})^{(3)} , \\ &H^{-2}(\Gamma, \Gamma) \simeq (\mathbb{Z}/2\mathbb{Z})^{(4)} \oplus \mathbb{Z}/2m\mathbb{Z} . \end{aligned}$$

Hence, as Proposition 3.1 says, $H^1(\Gamma, \Gamma)_{(l)} = H^{-1}(\Gamma, \Gamma)_{(l)} = 0$ and $H^0(\Gamma, \Gamma)_{(l)} = H^{-2}(\Gamma, \Gamma)_{(l)} \simeq \mathbb{Z}/l\mathbb{Z}$.

Next, we treat the case of $m \equiv 1 \pmod 4$. $\Lambda = \mathbb{Z}[(1 + \sqrt{m})/2]$ is a Frobenius $\mathbb{Z}$-algebra with $(1, (1 + \sqrt{m})/2)$, $((-1 + \sqrt{m})/2, 1)$ as the $\mathbb{Z}$-dual bases. We set $\Gamma = \Lambda \oplus \Lambda w_\sigma$, $w_\sigma^2 = \pm 1$; then $N_\Gamma(\Gamma) = m$. Note that $w_\sigma^\triangle = -w_\sigma$.

Proposition 3.2. *Notation and assumption being as above, let M be a Σ-module. Then $H^{-2}(\Gamma, \Gamma)$ has an invertible element in $H^*(\Gamma, \Gamma)$, hence $H^r(\Gamma, M)$ is periodic of period 2. In particular, $H^r(\Gamma, \Gamma) \simeq \mathbb{Z}/m\mathbb{Z}$ for r even, $=0$ for r odd.*

Proof. We will prove the proposition in a similar way as Proposition 3.1. In this case, we use the isomorphism:

$$\begin{aligned}
&H^{-1}(\Lambda, K(\Lambda)) \\
&\quad = (\Lambda / N_\Lambda(\Lambda))\left(1 \otimes \frac{1+\sqrt{m}}{2} - \frac{1+\sqrt{m}}{2} \otimes 1\right) \xrightarrow{\sim} \Lambda / N_\Lambda(\Lambda)
\end{aligned}$$

(see [7, Proposition 3.8]). Accordingly, $H^{-2}(\Lambda, \Gamma)$ has in invertible element u associated to $1 \otimes (1+\sqrt{m})/2 - (1+\sqrt{m})/2 \otimes 1$, and this implies $\tilde{\gamma}_\sigma(u) = u$. Let l be a prime integer such that $l \mid m$. By the proof of Theorem 2.4, $H^{-2}(\Gamma, \Gamma)_{(l)}$ has an invertible element in $H^*(\Gamma, \Gamma)_{(l)}$. While, since $N_\Gamma(\Gamma) = m$ and $2 \nmid m$, it follows from (2.3) that $H^{-2}(\Gamma, \Gamma)$ has an invertible element in $H^*(\Gamma, \Gamma)$. Hence $H^r(\Gamma, M)$ is periodic of period 2.

The statement in case of $r =$ even follows from

$$H^0(\Gamma, \Gamma) = \mathbb{Z}/N_\Gamma(\Gamma) = \mathbb{Z}/m\mathbb{Z}\,.$$

Now we have the following isomorphisms:

$$\begin{aligned}
&H^{-1}(\Lambda, \Gamma)_{(l)} \\
&\quad = ({}_{N_\Lambda}\Gamma / I_\Lambda(\Gamma))_{(l)} \\
&\quad = (\Lambda w_\sigma / \sqrt{m}(\Lambda w_\sigma))_{(l)} \\
&\quad = \left\{\left(\mathbb{Z} + \mathbb{Z}\,\frac{1+\sqrt{m}}{2}\right) w_\sigma \Big/ \left(\mathbb{Z}\sqrt{m} + \mathbb{Z}\,\frac{m+\sqrt{m}}{2}\right) w_\sigma\right\}_{(l)} \\
&\quad = \left\{\left(\mathbb{Z} + \mathbb{Z}\,\frac{m+\sqrt{m}}{2}\right) w_\sigma \Big/ \left(\mathbb{Z}m + \mathbb{Z}\,\frac{m+\sqrt{m}}{2}\right) w_\sigma\right\}_{(l)} \\
&\quad \simeq \mathbb{Z}/l\mathbb{Z}.
\end{aligned}$$

By (2.2),

$$\begin{aligned}
&\gamma_\sigma\left(\left(a + b\,\frac{m+\sqrt{m}}{2}\right) w_\sigma\right) \\
&\quad = w_\sigma^{-1}\left(\left(a + b\,\frac{m+\sqrt{m}}{2}\right) w_\sigma\right) w_\sigma^{\Delta} \\
&\quad = -\left(a + b\,\frac{m-\sqrt{m}}{2}\right) w_\sigma \\
&\quad = \left(-a - bm + b\,\frac{m+\sqrt{m}}{2}\right) w_\sigma\,.
\end{aligned}$$

This means $\gamma_\sigma = -\iota$ on $H^{-1}(\Lambda, \Gamma)_{(l)}$, hence, by Proposition 2.3, $H^{-1}(\Gamma, \Gamma)_{(l)} \simeq [H^{-1}(\Lambda, \Gamma)_{(l)}]^G = 0$. This implies that $H^{-1}(\Gamma, \Gamma)$ itself vanishes. $\square$

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